Literature Review

Raufs Dunamalijevs

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# 1 Annealing in Combinatorial Optimization

In this section we will discuss a class of metaheuristics that are inspired by the physical process of *annealing*. At the beginning, we will explain what it is. Then we will talk about multiple annealing metaheuristics that are relevant to this work. Finally, we will look at hardware named Digital Annealer.

## 1.1 Annealing

Annealing is a heat treatment process of matter like glass or metal (Collins, 1921; Wu and Fan, 2020). It involves heating up the material and then carefully cooling it down at a controlled rate. By following that process, it is possible to achieve a stable crystal structure.

If, however, this rate is exceeded, the resulting crystal will have defects as only locally optimal structures were allowed to form (Kirkpatrick, Gelatt and Vecchi, 1983). The lack of equilibrium will make such crystal metastable.

## 1.2 Simulated Annealing

Simulated annealing (SA) is a metaheuristic that uses the concept of temperature to approximate global optimum of a combinatorial optimization problem (Kirkpatrick, Gelatt and Vecchi, 1983; Bertsimas and Tsitsiklis, 1993) The temperature (*T*) is a variable that controls the likelihood of making locally non-optimal uphill moves. Just as in physical annealing, we start the optimization with a high temperature and decrease it at a controlled rate. If that rate is low enough, we are guaranteed to find a global optimum (Liang, Cheng and Lin, 2014). However, it is possible to find good approximations of the optima significantly faster at higher cooling rates.

The temperature concept allows the algorithm to explore the search space and find the optimal solution (Poole and Mackworth, 2017). The SA starts with a high temperature. At that point the algorithm acts like random walk, choosing random moves with high probability, which prevents us from being stuck at local optima – this is when exploration happens. At the end of the SA, when the temperature is low, the algorithm prioritizes advantageous moves. It acts like greedy algorithm and tries to find the best solution. As we shift from higher to lower temperatures, we gradually shift from exploring the search space to find the optimum.

Let *c* be current node, *n* be next node (neighbour of node *c*) and *evaluation(x)* be the function that evaluates the energy of the node *x*. Then the energy transition (*ΔE*) can be defined by the following formula:

*ΔE = evaluation(n) - evaluation(c)*

*ΔE* and *T* are the variables that will affect the probability of choosing node *n*, where *ΔE* is an energy transition (negative when minimising) that the proposed node will introduce, and *T* is controlling how *ΔE* influences the probability of choosing this node (Johnson *et al.*, 1989).

The *acceptance probability function* is the function that maps current node, proposed node, and Temperature to a probability. There are many functions that suit this purpose, the original being an exponential function (Kirkpatrick, Gelatt and Vecchi, 1983).

When maximizing, the negative sign should be removed from the formula.

If that acceptance probability is higher than a uniformly generated random number, then node *c* is rejected in favour of node *n*. However, when (downhill movement), we will always choose the proposed node (Kirkpatrick, Gelatt and Vecchi, 1983).

Pseudocode of SA (Johnson *et al.*, 1989) is given by:

c ← initial node

**while** T > threshold

n ← generate next node(c)

ΔE ← evaluate(n) – evaluate(c)

**if** ΔE ≤ 0 t

c ← n

**else if** P(c, n, T) ≥ random(0, 1)

c ← n

**end**

T ← cooling(T)

**end**

## 1.3 Quantum Annealing

Quantum annealing (QA) is a metaheuristic that uses quantum-mechanical fluctuations to find a global minimum of a function (Ohzeki and Nishimori, 2010; Tanaka and Tamura, 2012). In QA we start with a strong quantum field and gradually decrease its strength (quantum fluctuation) to find a ground state that is optimal. This is analogous to SA, where we decreased the temperature (thermal fluctuation) to achieve the same result.

QA begins in a uniform superposition of all possible states, but as we decrease the strength of quantum field, we approach a single solution state (Farhi *et al.*, 2001).

In SA neighbouring nodes are located 1 move away for the current node. In QA the neighbouring nodes are in a certain range. This range is defined by the strength of the quantum field, slowly ceasing in the process of annealing. Transition from one state to another can happen within that range. If there is a short wall between current state, which happens to be a local minimum, and the global minimum, SA algorithm will need to make a series of disadvantageous moves to reach a better solution or in other words *“climb”* over the wall. This is avoided in QA as there is a capacity to tunnel through short walls if the quantum field is strong enough, making QA faster compared to SA when dealing with search spaces with many thin barriers (Crosson and Harrow, 2016). This effect is called *quantum tunnelling*. Comparison of thermal hopping and quantum tunnelling can be seen on Figure 1.

Diagram

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Figure 1. Insert text

The fundamental feature and the main benefit of QA is that it rapidly samples a wide range of configurations when exploring the energy landscape (Higham and Bedford, 2021). Quantum Annealer has been developed and commercialised by D-wave (Johnson *et al.*, 2011) and is currently used to solve real-world problems (D-Wave Systems, 2021).

## 1.4 Digital Annealer

Digital Annealer (DA) is quantum-inspired custom CMOS hardware developed by Fujitsu Laboratories (Aramon *et al.*, 2019). Its algorithm is based on SA, but has 3 major differences:

1. Instead of considering a single neighbouring node and choosing it if the acceptance probability is higher than a uniform random number, DA considers all neighbouring nodes in parallel. If more than a single node was accepted, 1 of them will be chosen uniformly at random to move into. This is advantageous as in SA if the node was rejected, we need to generate and then consider a new one from the same position, which takes time. In DA, however, all the nodes are considered in parallel and the acceptance probability per “turn” is therefore higher.
2. DA uses *dynamic offset*, which means that if no nodes were accepted, the next acceptance probability will be artificially increased to help the algorithm overcome narrow barriers.
3. All runs begin with the same random state to save time (in SA that runs in parallel the initial states are generated individually for each of the runs). Otherwise, it would need to evaluate the new initial node and all the new neighbouring nodes for each run.

These differences can be observed in DA pseudocode (Aramon *et al.*, 2019):

initial state ← random state

**for** each run

current state ← initial state

Eoffset ← 0

**for** each step

update temperature if needed

**for** each neighbour *(in parallel)*

calculate ∆Eneighbour

consider neighbour using ∆Eneighbour - ∆Eoffset

if neighbour accepted, record

**end**

**if** accepted neighbours ≥ 1

choose one accepted neighbour uniformly at random

update current state and the energy landscape *(in parallel)*

Eoffset ← 0

**else**

increase Eoffset

**end**

**end**

**end**

Although it is hard to make comparisons as the technology is quickly developing, it has been shown that DA can be faster and produce more precise solutions compared to D-Wave 2000Q (penultimate generation of D-Wave’s QA devices) in certain scenarios (Ohzeki *et al.*, 2019). DA’s are used in industry and science (Fujitsu, no date; Snelling *et al.*, 2020)

Currently QA devices are expensive and difficult to run as the technology is still under development (Şeker *et al.*, 2020). D-Wave 2000Q also suffers from qubit noise that results in lower precision (Katzgraber and Novotny, 2018), and sparse connectivity between spins that makes it less effective (Hamerly *et al.*, 2018). These problems are actively worked on and some of them are addressed in more recent QA devices (Dattani, Szalay and Chancellor, 2019). As QA technology develops, it will likely outperform DA largely due to its quantum parallelism (Boyd, 2018).

# 2 QUBO

Quadratic Binary Unconstrained Optimization (QUBO) is a mathematical formulation that can be applied to many combinatorial optimization (CO) problems (Kochenberger *et al.*, 2014). Problems formulated in QUBO can be subsequently solved using QUBO solvers. QA devices developed by D-Wave can solve CO problems that are formulated as QUBO or Ising models (Cruz-Santos, Venegas-Andraca and Lanzagorta, 2019). It has been shown that QUBO and Ising models are equivalent (Lucas, 2014), therefore one can be derived from another. DA’s can only solve problems that are formulated in QUBO (Aramon *et al.*, 2019).

There are many combinatorial optimization problems from industry, government and science that can be reformulated in QUBO (Glover, Kochenberger and Du, 2019).

In this section we will discuss the structure of QUBO, basics of reformulating constrained CO problems into equivalent unconstrained QUBO models, algorithms that can be used to solve these problems and relevant programming libraries that can be useful when dealing with QUBO’s. We will introduce the concept of penalty, which will be explained in greater details in the next section.

## 2.1 QUBO model

QUBO models have been described in great details in the work of Glover, Kochenberger and Du (2019). This subsection will summarize the key information relevant to this project.

The QUBO model can be expressed in the following way:

where *y* is the value to be optimized for, *x* is a vector of decision variables and *Q* is a square matrix with coefficients. QUBO models are unconstrainted, the only restriction being that every variable in decision vector *x* should be either 0 or 1. It is self-contained as all the information needed for the optimisation is stored in the matrix Q. QUBO problems are NP-hard.

A simple example of how to convert a Boolean function into QUBO can be demonstrated with the following minimization problem:

Our function has 2 parts: linear and quadratic. As every *xj* belongs to {0, 1}, we can easily make the linear part quadratic (because 02=0 and 12=1), making the entire function quadratic.

We can then write it using matrixes in the form of , which is a QUBO model formula shown at the beginning.

## 2.2 Constraints and Penalties

In the previous subsection it has been shown how an unconstrained minimization problem can be reformulated as a QUBO. Constrained problems can be reformulated in this model too. The constraints will be expressed as quadratic penalties which will be added to the original objective function (Glover, Kochenberger and Du, 2019). The more constraints are broken, the larger is the overall penalty imposed. When none of the constraints are broken, the penalty imposed will be equal to 0.

Since we are trying to solve a minimization problem, solutions that brake constraints and impose penalty on the objective function will be avoided.

*M* is a positive penalty scalar (Verma and Lewis, 2020). It controls the effect that the broken constraints will have on the objective function. If we have a soft constraint that is allowed to be broken, the penalty scalar can be decreased, which will decrease the effect that the broken constraint will have on our function. It is possible to use multiple scalars for various constraints if they are of a different importance. But usually only a single *M* is used (Glover, Kochenberger and Du, 2019).

If the penalty coefficient is too low, the broken constraints will be undervalued, and the solution produced by the optimizer will be infeasible. On the other hand, if the penalty coefficient is too large, the solution process will be negatively impacted as the penalties will overwhelm the objective function making it harder to differentiate between good and bad solutions (Glover, Kochenberger and Du, 2019).

It is possible to use domain expertise to select an acceptable *M* in some cases (Glover, Kochenberger and Du, 2019), but it has also been shown that some problems can have solutions of better quality if you choose penalty coefficient carefully (Şeker, Tanoumand and Bodur, 2020). Finding an optimal penalty coefficient value is not trivial and different techniques have been proposed to estimate it (Verma and Lewis, 2020; Huang et al., 2021).

Quadratic penalties for certain constraint types are already known. Some of them are shown on the Table 1.

Table

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Table 1. CHANGE all the variables in the table are binary

Consider a problem where either x1 or x2 should be equal to 1, but they cannot be equal to 0 or 1 together. This constraint can be described with the following formula:

Using the penalty equivalent to our constraint, we can express our objective function in the following way:

where f(x) is the original objective function. We can see that the penalty is imposed only when both decision variables are equal to 0 or 1, that is when the constraint is broken.

## 2.3 Natural QUBO Formulation

There exist combinatorial optimization problems that can be naturally expressed as QUBO models. Some of them have been described by Glover, Kochenberger and Du (2019). One of such problems will be summarized in this subsection using their work to demonstrate the process of the ‘*natural*’ QUBO formulation.

### 2.3.1 Minimum Vertex Cover

Minimum Vertex Cover (MVC) is a CO problem, where we are given undirected graph with vertexes and edges represented by sets *V* and *E* respectively. A single vertex ‘covers’ the edges that it is incident to. *Vertex cover* is a subset of vertexes *V*, which covers all the edges *E*. Figure 2 compares vertex cover and standard subsets of *V*. In MVC we aim to find a vertex cover with the minimum number of vertices.

Diagram, schematic

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Figure 2. Visual representations of common subsets of V and a vertex cover

The decision vector *x* can be expressed in the following way. If vertex *j* is in the vertex cover, then is xj = 1. Otherwise, xj = 0. Then our minimization function, before we consider any constraints, can be defined as the sum of all vertices that are in the cover.

To get a feasible solution, we need to cover all the edges. This can be expressed as the following constraint: every edge should have at least 1 vertex that belongs to the vertex cover. Because if it does not, the edge under consideration is not covered and, therefore, not all the edges are covered. This is infeasible. Thus, the constrains are:

Table 1 contains a penalty equivalent to this type of constraints. Using it we can make the following minimization function:

This formula can be expressed as , where *c* – is a constant that has no influence on the optimisation process and can be removed. This leaves us with an unconstrained QUBO model.

### 2.3.2 Example Solution

We are given the following graph which we need to find an MVC for:

*A picture containing diagram

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Figure 3. Example MVC graph.

Using the formula derived previously, we can express and then transform the problem in the following way:

If we remove the constant *6M*, which has no effect on the optimization (we can add it after the optimization has finished if needed), from the end of the formula, and assign an arbitrary value of 8 to the penalty scalar *M*, we will get the following QUBO:

With the following solution:

Diagram

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Figure 4. Solution to the example MVC.

## 2.4 Non-Natural QUBO Formulation and Slack Variables

Not all CO problems have a natural QUBO formulation. Also, quadratic penalties for some constrains might be unknown. Such problems can be formulated in QUBO too. In the first part of this subsection, we will show a general approach to QUBO formulation. In the second part, we will show how any inequality constraint can be expressed as a quadratic penalty by introducing slack variables. Both processes have been accurately described by Glover, Kochenberger and Du (2019).

### 2.4.1 QUBO Formulation

Consider a constrained minimisation problem. Without quadratic penalties, it will have the following form:

where *x* is a binary decision vector and *C* is a square matrix. If the constraints of the problem consist of integers, they can always be written as (this will be explained in more details in the subsection denoted to slack variables):

Then the quadratic penalties that we need to add to the original objective function are:

And the resultant objective function with constraints will be:

Constant *c* can be removed as it has no effect on the optimization (it can be added after a solution is found if needed). That is how we get a standard QUBO model out of a general constrained problem that has no natural QUBO formulation.

### 2.4.2 Slack Variables

Consider the following inequality constraint:

It can be transferred into an equality constraint by adding a slack variable, *s.* The slack variable cannot be bigger than 7 because the lowest number that the left side of the inequality takes is -1 with (0,0,1) configuration. It also cannot be smaller than -3 as the largest number that the left side takes is 9 with (1,1,0) configuration.

If we express *s* using binary expansion, which will satisfy all configurations of decision variable *x*, we will get the following equality:

As both *xj* and *sj* are binary at this point, they can both be contained in a single vector: *x*. And all the coefficients can then be contained in a single matrix *A*.

Then, as shown in the previous part of this subsection, this constraint can be expressed as a quadratic penalty and subsequently used to achieve a standard QUBO model.

## 2.5 Algorithms for solving QUBOs

## 2.6 Relevant libraries

### 2.6.1 PyQUBO

### 2.6.2 qbsolv

# 3 Penalty Optimisation Techniques

In this section we will present multiple approaches to penalty optimisation.

https://www.sciencedirect.com/science/article/abs/pii/S1572528620300281#b3

## 3.1 Analytical

## 3.2 Numerical

## 3.3 Machine Learning

# 4 Summary

Summarize everything and form a research question (or repeat the one from project proposal).

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